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On the concepts of Lie and covariant derivatives of spinors: part III. Comparison with the invariant formalism

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Abstract. The general method for defining the Lie and covariant derivatives of a spinor field, presented in parts I and II, is compared with that provided by Buchdahl's 'invariant formalism'. The results are found to be essentially in agreement (where the invariant formalism is applicable).

1. Introduction

In this series of articles, we are considering the problem of covariant derivatives of spinor fields in spaces with a general connection. (This work can also be interpreted as a study of Lie differentiation of spinor fields along arbitrary vector fields.) In part I, we presented our definition [1] of the covariant derivative, and in part II we analysed it [2]. In particular, it was emphasized how our formalism is compatible with tensor calculus in full generality, i.e. without restriction on the spacetime connection, whereas other frameworks (e.g. [3–5]) are confined to special cases.

When the spacetime connection happens to be conformal, a framework different from all the previous ones was introduced by Buchdahl [6, 7] in 1992. It uses the concept of 'gauge and phase invariance' of a spinor field, and will therefore be referred to here as the 'invariant formalism'. Given that our definition is valid in full generality, it should be possible to relate it, in the conformal case, to the invariant formalism. The present article, part III, is devoted to investigating this relationship.

We shall see that the 'gauge and phase weights' of the invariant formalism are closely related to the parameter k appearing in our framework. (See (2.2) of part II.) Moreover, it will become clear that, if the weights are suitably chosen, the invariant formalism is equivalent to ours (in the special case where the invariant formalism is applicable). These considerations will be presented in section 3.

The invariant formalism also introduces a particular modification of the definition of the covariant derivative of tensors, whereas we retain the conventional tensorial definition and construct a definition of the covariant derivative of spinor fields compatible with the conventional tensorial derivative. (Details are found in section 5 of part II.) This important difference sheds light on the invariant formalism, and this question will be examined in section 4.

In order to recast the invariant formalism into our general framework and notation, we shall also present, in section 2, a summary of the invariant formalism. We shall emphasize the aspects which will be of relevance for our later considerations.

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2. The invariant formalism

The purpose of the invariant formalism [7] is to introduce definitions of the covariant derivatives of tensors and spinors which are consistent and convenient in spaces having a conformal connection, i.e. in Weyl spaces. (For details of the motivation for the invariant formalism, see [7].) Weyl spaces are special cases of those considered in parts I and II, and therefore it is useful to recall here, in the special case, the general definitions of part I.

An arbitrary connection Γ defined on a manifold exhibits in general torsion (T), curvature (R), and non-metricity (H). If ∇, $e_{(\mu)}$, and $e^{(\mu)}$ denote, respectively, the covariant derivative, an arbitrary basis in the tangent space and the dual of this basis, one may define the relevant quantities as

$$\begin{aligned} T(X,Y) &\equiv \nabla_X Y - \nabla_Y X - [X,Y] \\ &\equiv T^{\gamma}{}_{\alpha\beta} e^{(\alpha)} \otimes e^{(\beta)} \otimes e_{(\gamma)} \end{aligned} \tag{2.1}$$

$$\begin{aligned} H(Z,X,Y) &\equiv (\nabla_Z g)(X,Y) \\ &\equiv H_{\alpha\beta\gamma} e^{(\alpha)} \otimes e^{(\beta)} \otimes e^{(\gamma)} \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\nabla_\alpha e_{(\beta)} \equiv \nabla_{e_{(\alpha)}} e_{(\beta)} \equiv \Gamma^{\gamma}{}_{\beta\alpha} e_{(\gamma)} \\ &[e_{(\alpha)}, e_{(\beta)}] \equiv D^{\gamma}{}_{\alpha\beta} e_{(\gamma)} \end{aligned} \tag{2.3}$$

where X, Y, Z are vector fields and g is the metric. (The definition of the curvature R will not be necessary for what follows and is therefore not reproduced here.) These relations yield the explicit expression of the connection Γ in terms of g, T, H and D:

$$\begin{aligned} &\Gamma_{\alpha\beta\gamma} \equiv \langle\alpha\beta\gamma\rangle + Q_{\alpha\beta\gamma} - K_{\alpha\beta\gamma} \\ &\langle\alpha\beta\gamma\rangle \equiv [\alpha\beta\gamma] + C_{\alpha\beta\gamma} \qquad \text{(Levi-Civita or Riemannian connection)} \\ &[\alpha\beta\gamma] \equiv \Sigma(e_{(\gamma)}(g_{\alpha\beta})) = +[\alpha\gamma\beta] \qquad \text{(Christoffel symbol)} \\ &C_{\alpha\beta\gamma} \equiv \Sigma(D_{\gamma\alpha\beta}) = -C_{\beta\alpha\gamma} \qquad \text{(non-holonomicity)} \\ &Q_{\alpha\beta\gamma} \equiv \Sigma(T_{\gamma\alpha\beta}) = -Q_{\beta\alpha\gamma} \qquad \text{(contorsion tensor)} \\ &K_{\alpha\beta\gamma} \equiv \Sigma(H_{\gamma\alpha\beta}) = +K_{\alpha\gamma\beta} \qquad \text{(non-metric part)} \end{aligned} \tag{2.4}$$

in which the symbol Σ applied to a three-index object $W_{\gamma\alpha\beta}$ is defined as

$$2\Sigma(W_{\gamma\alpha\beta}) \equiv W_{\gamma\alpha\beta} + W_{\beta\alpha\gamma} - W_{\alpha\beta\gamma}. \tag{2.5}$$

(In the natural basis $e_{(\mu)} \equiv \partial/\partial x^{\mu}$, the commutation coefficients D vanish, and so does the non-holonomicity C; on the other hand, in an orthonormal frame $e_{(\hat{\mu})}$, the metric is constant and the Christoffel symbol $[\hat{\alpha}\hat{\beta}\hat{\gamma}]$ vanishes.) It is important to emphasize that, to distinguish tensorial components in an orthonormal frame from those in an arbitrary one, the former will be surmounted by a caret. Thus, $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}}$ denotes the orthonormal components of the connection, whereas $\Gamma_{\alpha\beta\mu}$ denotes arbitrary components.

The various terms in the connection Γ of (2.4) give rise to different covariant derivatives: $\langle\alpha\beta\gamma\rangle$ generates the Riemannian (i.e. metric-compatible and torsion-free) covariant derivative, $\tilde{\Gamma}_{\alpha\beta\gamma} \equiv \langle\alpha\beta\gamma\rangle + Q_{\alpha\beta\gamma}$ generates the Riemann–Cartan (i.e. metric-compatible) covariant derivative (with torsion), and the complete connection Γ takes into account both the torsion T and the non-metricity H. With the above conventions, they read, as exemplified on the tensor $t \equiv t^{\mu}{}_{\nu} e^{(\nu)} \otimes e_{(\mu)}$:

$$\begin{aligned}
\nabla_{\rho} t &\equiv t^{\mu}{}_{\nu|\rho} e^{(\nu)} \otimes e^{(\rho)} \otimes e_{(\mu)} \\
t^{\mu}{}_{\nu|\rho} &= e_{(\rho)}(t^{\mu}{}_{\nu}) + \Gamma^{\mu}{}_{\sigma\rho} t^{\sigma}{}_{\nu} - \Gamma^{\sigma}{}_{\nu\rho} t^{\mu}{}_{\sigma} \\
\tilde{\nabla}_{\rho} t &\equiv t^{\mu}{}_{\nu:\rho} e^{(\nu)} \otimes e^{(\rho)} \otimes e_{(\mu)} \\
t^{\mu}{}_{\nu:\rho} &= e_{(\rho)}(t^{\mu}{}_{\nu}) + \tilde{\Gamma}^{\mu}{}_{\sigma\rho} t^{\sigma}{}_{\nu} - \tilde{\Gamma}^{\sigma}{}_{\nu\rho} t^{\mu}{}_{\sigma} \\
{}^{0}\nabla_{\rho} t &\equiv t^{\mu}{}_{\nu!\rho} e^{(\nu)} \otimes e^{(\rho)} \otimes e_{(\mu)} \\
t^{\mu}{}_{\nu!\rho} &= e_{(\rho)}(t^{\mu}{}_{\nu}) + \langle^{\mu}{}_{\sigma\rho}\rangle t^{\sigma}{}_{\nu} - \langle^{\sigma}{}_{\nu\rho}\rangle t^{\mu}{}_{\sigma}.
\end{aligned} \tag{2.6}$$

Weyl spaces are defined as manifolds in which the non-metricity $H_{\mu\alpha\beta} \equiv g_{\alpha\beta|\mu}$ takes the special form $H_{\mu\alpha\beta} = 2g_{\alpha\beta}A_{\mu}$, where A_{μ} is a covariant vector field called the Weyl vector. The formalism (2.4) simplifies then, and $K_{\alpha\beta\gamma}$ reads

$$K_{\alpha\beta\gamma} = g_{\alpha\beta}A_{\gamma} + g_{\alpha\gamma}A_{\beta} - g_{\beta\gamma}A_{\alpha}. \tag{2.7}$$

In particular, the covariant derivatives of the metric tensor with respect to the operators ∇ and $\tilde{\nabla}$ of (2.6) are related by

$$g_{\mu\nu|\rho} = g_{\mu\nu:\rho} + 2g_{\mu\nu}A_{\rho} \tag{2.8}$$

which will play an important role in the conclusion.

In the context of Weyl spaces, a gauge transformation is defined as the following operation acting on the fields $g_{\mu\nu}$, $T_{\mu\nu\rho}$, A_{μ}, $D_{\mu\nu\rho}$, and generated by a scalar field Φ:

$$\begin{aligned}
g'_{\mu\nu} &\equiv e^{2\Phi} g_{\mu\nu} \\
T'_{\mu\nu\rho} &\equiv e^{2\Phi} T_{\mu\nu\rho} \\
A'_{\mu} &\equiv A_{\mu} + e_{(\mu)}(\Phi) \\
D'_{\mu\nu\rho} &\equiv e^{2\Phi} D_{\mu\nu\rho}.
\end{aligned} \tag{2.9}$$

(It should be noted that, originally, Weyl spaces were assumed to be torsion-free and were developed in the language of natural frames only. These restrictions, however, are unnecessary.)

One proves easily from (2.4) and (2.7) that, under a gauge transformation, the connection Γ and the non-metricity transform as

$$\begin{aligned}
\Gamma'_{\alpha\beta\gamma} &= e^{2\Phi} \Gamma_{\alpha\beta\gamma} \\
\Gamma'^{\alpha}{}_{\beta\gamma} &= 1\Gamma^{\alpha}{}_{\beta\gamma} \\
g'_{\mu\nu|\rho} &= e^{2\Phi}[g_{\mu\nu|\rho} + 2g_{\mu\nu}e_{(\rho)}(\Phi)].
\end{aligned} \tag{2.10}$$

The transformation (2.10) of the non-metricity implies that, if one defines a new covariant derivative, denoted by a semi-colon, as

$$g_{\mu\nu;\rho} \equiv g_{\mu\nu|\rho} - 2g_{\mu\nu}A_{\rho} \tag{2.11}$$

it will satisfy

$$g'_{\mu\nu;\rho} = e^{2\Phi} g_{\mu\nu;\rho}. \tag{2.12}$$

This is the fundamental idea of the invariant formalism [7]: one may decide to abandon the conventional covariant derivative | of a tensor $t = t^{\mu}{}_{\nu} e^{(\nu)} \otimes e_{(\mu)}$, introduced in (2.6), and replace it by the 'gauge-invariant' covariant derivative

$$\begin{aligned} t^{\mu}{}_{\nu;\rho} &\equiv t^{\mu}{}_{\nu|\rho} - nA_{\rho} t^{\mu}{}_{\nu} \\ &= e_{(\rho)}(t^{\mu}{}_{\nu}) + \Gamma^{\mu}{}_{\sigma\rho} t^{\sigma}{}_{\nu} - \Gamma^{\sigma}{}_{\nu\rho} t^{\mu}{}_{\sigma} - nA_{\rho} t^{\mu}{}_{\nu} \end{aligned} \tag{2.13}$$

in which n is the 'weight' of t, namely the number such that the following holds under a gauge transformation:

$$t'^{\mu}{}_{\nu} = e^{n\Phi} t^{\mu}{}_{\nu}. \tag{2.14}$$

(Different tensors have, of course, different weights.) It follows from (2.13) and (2.14) that $t'^{\mu}{}_{\nu;\rho}$ satisfies

$$t'^{\mu}{}_{\nu;\rho} = e^{n\Phi} t^{\mu}{}_{\nu;\rho} \tag{2.15}$$

which generalizes (2.12) to a tensor different from the metric.

It is permissible to interpret (2.13) as saying that the appropriate covariant derivative of a tensor t in a Weyl space is the normal one minus a term proportional to the Weyl vector and to the tensor t undergoing differentiation. In other words, one may consider, as is done in the invariant formalism, that the non-metricity affects the covariant derivative by subtraction of a term proportional to t and function of the Weyl vector. Similarly, one may adopt this procedure for spinors, and write for a two-component spinor $u \equiv u_a \tilde{e}^{(a)}$ the covariant derivative in a Weyl space as the normal one minus a term proportional to u:

$$u_{a;\mu} \equiv e_{(\mu)}(u_a) - \Gamma^{b}{}_{a\mu} u_b - (nA_{\mu} + \mathrm{i}\nu\kappa_{\mu}) u_a \tag{2.16}$$

where a component notation is understood as in (2.13), $\Gamma^{b}{}_{a\mu}$ is the spin connection (defined in [7] and analysed in section 3), and (n, ν) is the 'weight' of u_a, i.e. the couple of numbers appearing in the transformation law

$$u'_a = e^{(n\Phi + \mathrm{i}\nu\Psi)} u_a \tag{2.17}$$

which defines a 'gauge and phase transformation' of a spinor u. By analogy with the behaviour of the Weyl vector A in (2.9), κ_{μ} is assumed to satisfy

$$\kappa'_{\mu} = \kappa_{\mu} + e_{(\mu)}(\Psi). \tag{2.18}$$

A detailed discussion on how the weight (n, ν) of a spinor u_a is determined by the invariant formalism can be found in [7]. What we are going to investigate here, in section 3, is whether there exists a choice for (n, ν) which makes (2.16) identical to our definition of the covariant derivative of a spinor introduced in parts I and II.

It is important to emphasize that, by adopting (2.13) as the definition of the tensorial covariant derivative, as opposed to the conventional one denoted by | in (2.6), a radical departure has been taken from tensor calculus. The computational advantages of this change are outlined in [7], one of them being the covariant constancy of $g_{\mu\nu}$ with respect to the new covariant derivative, but computational ease is not our point of view here. In part II of our formalism, we proved that *our* definition of the covariant derivative of a spinor field is compatible with *conventional* tensor calculus. It is therefore important to investigate in more detail the status of the invariant formalism, which employs a *modified* version of tensor calculus. This will be done in section 4 after establishing, in section 3, that the definition (2.16) for spinors is essentially equivalent to ours of parts I and II.

3. The spinorial covariant derivative of parts I and II compared with that of the invariant formalism

In part II, we considered the following definition of the covariant derivative $\nabla_X u$ of the two-component spinor $u \equiv u^a \tilde{e}_{(a)}$ along the vector X:

$$\begin{aligned} &\nabla_X u \equiv [X(u^m) - \tilde{\mathcal{A}}^m{}_n(X)u^n]\tilde{e}_{(m)} \\ &-2\tilde{\mathcal{A}}^m{}_n(X) \equiv {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n + \tfrac{1}{4}k\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\delta^m{}_n \\ &(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n \equiv \sigma^{[\hat{\mu}|\dot{p}m|}\sigma^{\hat{\nu}]}{}_{\dot{p}n} \end{aligned} \tag{3.1}$$

where ${}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ and ${}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ denote, respectively, the antisymmetric and the symmetric parts of the spacetime connection $\mathcal{A}_{\hat{\mu}\hat{\nu}}$ in an orthonormal frame $e_{(\hat{\alpha})}$, k is a free parameter, and $\sigma^{\hat{\mu}\dot{a}}{}_a$ is the Infeld–van der Waerden symbol [8, 9]. The notation is the same as in part II. In particular, the antisymmetric and the symmetric parts ${}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ and ${}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ of the connection are given in terms of the metric g, the contorsion Q, the non-holonomicity C, and the non-metricity H by

$$\begin{aligned} &2\,{}^{\mathrm{S}}\mathcal{A}_{\alpha\beta} \equiv (\Gamma_{\alpha\beta\mu} + \Gamma_{\beta\alpha\mu})e^{(\mu)} = [e_{(\mu)}(g_{\alpha\beta}) - H_{\mu\alpha\beta}]e^{(\mu)} \\ &{}^{\mathrm{A}}\mathcal{A}_{\alpha\beta} \equiv \tfrac{1}{2}(\Gamma_{\alpha\beta\mu} - \Gamma_{\beta\alpha\mu})e^{(\mu)} = \{-e_{([\alpha)}(g_{\beta]\mu}) + C_{\alpha\beta\mu} + Q_{\alpha\beta\mu} + H_{[\alpha\beta]\mu}\}e^{(\mu)} \end{aligned} \tag{3.2}$$

which will be necessary later.

In order to compare (3.1), (3.2) with the covariant derivative (2.16) provided by the invariant formalism, it is sufficient to restrict attention to the derivative along a vector field X which is a basic vector, say $e_{(\hat{\mu})}$. We obtain thus, by (3.1) and (3.2),

$$\begin{aligned} &\nabla_{\hat{\alpha}} u \equiv \nabla_{e_{(\hat{\alpha})}} u \equiv u^m{}_{||\hat{\alpha}}\tilde{e}_{(m)} \\ &u^m{}_{||\hat{\alpha}} \equiv e_{(\hat{\alpha})}(u^m) - \tilde{\mathcal{A}}^m{}_{n\hat{\alpha}}u^n \\ &-4\tilde{\mathcal{A}}^m{}_{n\hat{\alpha}} \equiv (\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} - \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}})(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n + \tfrac{1}{4}k\eta^{\mu\nu}(\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} + \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}})\delta^m{}_n \\ &\qquad = 2\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n - \tfrac{1}{4}kH_{\hat{\alpha}}{}^{\hat{\beta}}{}_{\hat{\beta}}u^m \end{aligned} \tag{3.3}$$

where the last line follows from the antisymmetry of $\sigma^{\hat{\mu}\hat{\nu}}$, and the notation $||$ has been adopted for the covariant derivative of a spinor (in components), in order to distinguish it from the derivative $|$ which applies to tensors. When, in (3.3), the connection Γ is substituted in terms of the metric-compatible connection $\tilde{\Gamma}$ and the non-metricity H, as in (2.4), the result reads

$$u^m{}_{||\hat{\alpha}} = e_{(\hat{\alpha})}(u^m) + \tfrac{1}{2}\tilde{\Gamma}_{\hat{\mu}\hat{\nu}\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n u^n + \tfrac{1}{2}H_{[\hat{\mu}\hat{\nu}]\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n u^n - \tfrac{1}{16}kH_{\hat{\alpha}}{}^{\hat{\beta}}{}_{\hat{\beta}}u^m. \tag{3.4}$$

So far, all the equations are valid for a general connection. In the special case of a conformal connection, the non-metricity takes the value

$$H_{\hat{\mu}\hat{\alpha}\hat{\beta}} = 2A_{\hat{\mu}}\eta_{\alpha\beta} \tag{3.5}$$

in an orthonormal frame, which enables one to simplify (3.4) as

$$u^m{}_{||\hat{\alpha}} = e_{(\hat{\alpha})}(u^m) + \tfrac{1}{2}\tilde{\Gamma}_{\hat{\mu}\hat{\nu}\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^m{}_n u^n + A^{\hat{\mu}}(\sigma_{\hat{\mu}\hat{\alpha}})^m{}_n u^n - \tfrac{1}{2}kA_{\hat{\alpha}}u^m. \tag{3.6}$$

We are now ready to compare (3.6) with the definition (2.16) of the invariant formalism.

By virtue of (2.16), the covariant derivative of a spinor $v \equiv v_a \tilde{e}^{(a)}$ of weight (n, ν) reads, according to the invariant formalism,

$$\begin{aligned} v_{a;\mu} &= e_{(\mu)}(v_a) - \Gamma^b{}_{a\mu} v_b - M_\mu v_a \\ M_\mu &\equiv n A_\mu + \mathrm{i}\nu\kappa_\mu . \end{aligned} \tag{3.7}$$

The spin connection $\Gamma^b{}_{a\mu}$ is found by the invariant formalism [7] to be given in terms of the metric-compatible spin connection $\tilde{\Gamma}^b{}_{a\mu}$ by

$$\begin{aligned} \Gamma^b{}_{a\mu} &= \tilde{\Gamma}^b{}_{a\mu} + \Delta^b{}_{a\mu} \\ \Delta^b{}_{a\mu} &\equiv A^\nu \sigma_{[\nu}{}^{\dot{c}b}\sigma_{\mu]\dot{c}a} - \tfrac{1}{2}\delta^b{}_a K_\mu \\ K_\mu &\equiv z A_\mu + \mathrm{i}\zeta\kappa_\mu \end{aligned} \tag{3.8}$$

where z and ζ denote, respectively, the gauge weight and the phase weight of the spinor metric. (In part II, the spinor metric, expressed in a spin frame $\tilde{e}^{(a)}$ above an orthonormal frame $e^{(\hat{\mu})}$, had its components denoted by ϵ_{ab}.) Therefore, as a result of (3.8), the covariant derivative (3.7) becomes

$$v_{a;\mu} = e_{(\mu)}(v_a) - \tilde{\Gamma}^b{}_{a\mu} v_b - A^\alpha \sigma_{[\alpha}{}^{\dot{c}b}\sigma_{\mu]\dot{c}a} v_b + \left(\tfrac{1}{2}K_\mu - M_\mu\right) v_a . \tag{3.9}$$

Moreover, in the invariant formalism [6], equation (3.4), the metric-compatible spin connection $\tilde{\Gamma}^b{}_{a\mu}$ has the expression

$$\begin{aligned} 2\tilde{\Gamma}^b{}_{a\mu} &= \sigma^{\alpha\dot{c}b} e_{(\mu)}(\sigma_{\alpha\dot{c}a}) - \tilde{\Gamma}^\alpha{}_{\beta\mu}\sigma^{\beta\dot{c}b}\sigma_{\alpha\dot{c}a} - \tfrac{1}{2}\delta^b{}_a \bar{\lambda}_{,\mu} \\ \lambda_{,\mu} &\equiv e_{(\mu)}[\log|\det(\gamma_{ab})|] \end{aligned} \tag{3.10}$$

in which $\tilde{\Gamma}^\alpha{}_{\beta\mu}$ and γ_{ab} are, respectively, the metric-compatible part of the spacetime connection and the spinor metric in an arbitrary frame. Let us now use an orthonormal basis $e_{(\hat{\mu})}$ to simplify the calculations.

In an orthonormal frame, both the spinor metric and the Infeld–van der Waerden symbols are constant, and therefore the metric-compatible spin connection $\tilde{\Gamma}^b{}_{a\hat{\mu}}$ of (3.10) becomes

$$-2\tilde{\Gamma}^b{}_{a\hat{\mu}} = \tilde{\Gamma}^{\hat{\alpha}\hat{\beta}}{}_{\hat{\mu}}\sigma_{\hat{\beta}}{}^{\dot{c}b}\sigma_{\hat{\alpha}\dot{c}a} = \tilde{\Gamma}^{\hat{\alpha}\hat{\beta}}{}_{\hat{\mu}}\sigma_{[\hat{\beta}}{}^{\dot{c}b}\sigma_{\hat{\alpha}]\dot{c}a} = -\tilde{\Gamma}^{\hat{\alpha}\hat{\beta}}{}_{\hat{\mu}}(\sigma_{\hat{\alpha}\hat{\beta}})^b{}_a \tag{3.11}$$

in which the antisymmetry of the metric-compatible spacetime connection in an orthonormal frame has been employed, as well as the definition (3.1) of $\sigma_{\hat{\alpha}\hat{\beta}}$. The covariant derivative (3.9) simplifies then and reads

$$v_{a;\hat{\mu}} = e_{(\hat{\mu})}(v_a) - \tfrac{1}{2}\tilde{\Gamma}_{\hat{\alpha}\hat{\beta}\hat{\mu}}(\sigma^{\hat{\alpha}\hat{\beta}})^b{}_a v_b - A^{\hat{\alpha}}(\sigma_{\hat{\alpha}\hat{\mu}})^b{}_a v_b + \left(\tfrac{1}{2}K_{\hat{\mu}} - M_{\hat{\mu}}\right) v_a . \tag{3.12}$$

In this form, the parallel with our definition (3.6) is obvious. The apparent discrepancy in sign between (3.6) and (3.12) arises from the fact that (3.12), based on [7], applies to the *covariant* spinor $v = v_a \tilde{e}^{(a)}$, whereas (3.6) applies to the *contravariant* spinor $u = u^a \tilde{e}_{(a)}$. There is thus no conflict. The only difference comes from the conformal term $kA_{\dot{\alpha}}u^m/2$ in

(3.6) and $(K_{\hat{\mu}}/2 - M_{\hat{\mu}})v_a$ in (3.12). These last contributions also agree with one another provided

$$K_{\hat{\mu}} - 2M_{\hat{\mu}} = kA_{\hat{\mu}} \tag{3.13}$$

which implies, after using the definitions (3.7) and (3.8) for $M_{\hat{\mu}}$ and $K_{\hat{\mu}}$, that the weights (n, ν) of v_a and (z, ζ) of the spinor metric must satisfy

$$(2n - z + k)A_{\hat{\mu}} + (2\nu - \zeta)\mathrm{i}\kappa_{\hat{\mu}} = 0. \tag{3.14}$$

For this equality to hold irrespectively of A and κ, the necessary and sufficient condition on the weight (n, ν) of v_a reads as

$$2(n, \nu) = (z - k, \zeta). \tag{3.15}$$

Consequently, if the weight (n, ν) is considered as *determined* by (3.15), as opposed to being imposed *a priori* by the requirement (2.17) of gauge and phase invariance, the result (3.15) proves that it is always possible to ensure that the invariant formalism agrees with our definition (3.6). On the other hand, if one considers the weight as *fixed* by the invariant formalism, then (3.15) proves that the invariant formalism agrees with ours *only* for spinors of that particular weight (n, ν) given by (3.15). The consequences of this observation will now be analysed.

4. Conclusion

In this article, we compared the definition of the covariant derivative of a spinor provided by our framework [1, 2] with that of the invariant formalism [7]. Each of these formalisms has an advantage over the other: Our method has the advantage of being valid in full generality, i.e. without restriction on the spacetime connection; on the other hand, the invariant formalism, valid only in Weyl spaces, is *computationally* very convenient since it leads to the adoption of alternative tensorial and spinorial covariant derivatives, (2.13) and (2.16), with respect to which the metric tensor and the spinor metric are covariantly constant, even in spite of the non-metricity present in Weyl spaces. This feature is one of the main reasons for introducing the invariant formalism. (See [7] for details.) From our point of view, however, computational convenience is not a decisive factor.

Light is shed on the alternative tensorial covariant derivative of the invariant formalism by returning to the relationship between the conventional tensorial covariant derivative | and the (conventional) metric-compatible one : of (2.8). If one solves (2.8) for the metric-compatible covariant derivative, one finds

$$g_{\mu\nu:\rho} = g_{\mu\nu|\rho} - 2g_{\mu\nu}A_\rho. \tag{4.1}$$

When compared with (2.11), this shows that the covariant derivative of the metric according to the invariant formalism coincides with the conventional metric-compatible derivative. It comes, therefore, as no surprise that the invariant formalism obtains a covariantly constant metric, since its notion of covariant derivative, when applied to the metric, is nothing more than the conventional metric-compatible part : of the total covariant derivative |. This last property, however, holds only for the derivative of the metric. In other words, for a general tensor, it is not true that $t^{\mu}{}_{\nu;\rho} = t^{\mu}{}_{\nu:\rho}$, as it follows from (2.4), (2.6), and (2.13) since the term $nA_{\rho}t^{\mu}{}_{\nu}$ does, in general, *not* drop out, so that ; and : are genuinely inequivalent.

According to the result (3.15), it is always possible to choose the system of weights so that the invariant formalism agrees with ours, specialized to a Weyl space. It is important to emphasize, however, that our formalism has been proved [2] consistent with *conventional* tensor calculus in a general space, and thus in a Weyl space in particular. Therefore, when weights are selected according to (3.15), compatibility is *not* achieved with respect to the *alternative* tensorial covariant derivative (2.13). For consistency with the alternative definition (2.13), the weights must be selected according to the rules of the invariant formalism. (Details are available in [7].) When the weights are then considered as fixed by the invariant formalism, the gauge- and phase-invariant covariant derivative no longer agrees with ours for all types of spinors, but only for that subclass of spinors with weight (n, ν) that happens to satisfy (3.15).

This creates no difficulty or confusion: in Weyl spaces, one may either adopt, for computational ease, the invariant formalism, at the expense of having to use a modified version of tensor calculus; or one may opt for our formalism, which retains conventional tensor calculus. It is possible also, in principle, to 'mix' the formalisms since the constraint (3.15) gives the necessary and sufficient condition for the 'mixture' to be self-consistent. Little, however, seems to be gained in such a 'mixture', which loses the advantages of either formalism.

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References

[1] Hurley D and Vandyck M 1994 *J. Phys. A: Math. Gen.* **27** 1743

[2] Hurley D and Vandyck M 1994 *J. Phys. A: Math. Gen.* **27** 5941

[3] Penrose R and Rindler 1987 *Spinors and Space-time* vol 1 (Cambridge: Cambridge University Press) preface, chs 3.1, 3.2

[4] Benn I and Tucker R 1988 *An Introduction to Spinors and Geometry* (Bristol: Adam Hilger) chs 2.8, 2.4, 2.5

[5] Huggett S and Tod K 1985 *An Introduction to Twistor Theory* (Cambridge: Cambridge University Press) ch 5

[6] Buchdahl H 1989 *J. Math. Phys.* **30** 700

[7] Buchdahl H 1992 *J. Math. Phys.* **33** 64

[8] van der Waerden B 1929 *Nachr. Akad. Wiss. Götting. Math.-Phys.* K 1 100

[9] Infeld L and van der Waerden B 1933 *Sitz. Ber. Preuss. Akad. Wiss. Phys.-Math.* K **1** 380